200. Proposed by R. D. CARMICHAEL, Indiana University.

Find the general solution, in relatively prime integers, of the equation $x^2 + y^2 = z^4$.

201. Proposed by E. T. BELL, Seattle, Washington.

Eisenstein (*Crelle*, t. 27, p. 282) proposed, as the simplest of several problems: "In the expansion of

$$\frac{1 + z + z^2 + \cdots + z^{p-1}}{(1 - z)^{p-1}} - 1,$$

where p is prime, to show that the coefficients of the various powers of z are all divisible by p."

202. Proposed by A. R. SCHWEITZER, Chicago, Ill.

There exists an infinitude of systems of dyads $\{\alpha\beta\}$ in 7, 9, 11, etc., elements such that each system has the following properties: (1) if $\alpha\beta$ is in the set, then $\beta\alpha$ is not in the set; (2) for each dyad $\alpha\beta$ in the set there exists an element ξ such that $\xi\beta$ and $\alpha\xi$ are also in the set. For example, such a system is,

12,	23,	34,	45,	56,	67,	78,	89,	91
13,	24,	35,	46,	57,	68,	79,	81,	92
14,	25,	36,	47,	58,	69,	71,	82,	93
51,	62,	73,	84,	95,	16,	27,	38,	49

Investigate the existence of

I. A finite set of triads $\{\alpha\beta\gamma\}$ such that (1) if $\alpha\beta\gamma$ is in the set, then $\beta\gamma\alpha$, $\gamma\alpha\beta$ are also in the set but $\beta\alpha\gamma$ is not in the set, (2) for each triad $\alpha\beta\gamma$ in the set there exists an element ξ such that $\xi\beta\gamma$, $\alpha\xi\gamma$, $\alpha\beta\xi$ are also in the set.

II. A finite set of tetrads $\{\alpha\beta\gamma\delta\}$ such that (1) if $\alpha\beta\gamma\delta$ is in the set, then $\beta\gamma\alpha\delta$, $\gamma\alpha\beta\delta$, $\gamma\delta\alpha\beta$ are also in the set but $\beta\alpha\gamma\delta$ is not in the set, (2) for each tetrad $\alpha\beta\gamma\delta$ in the set there exists an element ξ such that $\xi\beta\gamma\delta$, $\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi$ are also in the set.

The problem for alternating *n-ads* for $n > 4$ is obvious.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

382. Proposed by C. E. FLANAGAN, Wheeling, W. Va.

A few days ago, I deduced the following formula for finding the value of the unknown quantity in a cubic equation having the form, $x^3 + 3A^2x = B$.

Let $C = \sqrt{\dfrac{2B}{A^3} + 1} - 1$. Then $x = \dfrac{2B}{A^2(C^2 + 12)} + \dfrac{AC}{4}$.

It is required to show: (1) How this formula was derived; (2) under what conditions does it give exact results; (3) in general, what is its degree of approximation; (4) if possible, modify it so that it will always give exact results; (5) if it cannot be so modified, show why.

SOLUTION BY THE PROPOSER.

(1) The formula was derived by equating the volume of a spherical segment of one base and unknown altitude, x, to n times the volume of a sphere whose radius is the radius of the base of the spherical segment.

Thus, let r be the given radius of the base of the spherical segment of one base and x the unknown altitude. Then we have by hypothesis

$$\tfrac{1}{3}\pi x^2(3r - x) = n(\tfrac{4}{3}\pi r^3) = \tfrac{1}{2}\pi x r^2 + \tfrac{1}{6}\pi x^3.$$

Hence,

$$3rx^2 - x^3 = 4nr^3,$$

and

$$3r^2x + x^3 = 8nr^3;$$

whence, by adding the two equations and reducing,

$$r^2x + rx^2 = 4nr^3.$$

Whence

$$x = -\frac{r}{2} \pm \frac{r}{2}\sqrt{16n + 1}.$$

Letting $r^2 = A^2$, $B = 8nr^3$, and $C = \sqrt{\frac{2B}{A^3} + 1} - 1$, we have

(1) $$x = \frac{AC}{2}.$$

But

$$\frac{B}{\frac{A^3C^3}{8} + \frac{3A^3C}{2}} = 1.$$

Hence,

(2) $$x = \frac{AC}{2} \times \frac{B}{\frac{A^3C^3}{8} + \frac{3A^3C}{2}} = \frac{4B}{A^2(C^2 + 12)}.$$

From the last two values of x, we have, by addition and dividing by 2,

(3) $$x = \frac{2B}{A^2(C^2 + 12)} + \frac{AC}{4}.$$

(2) This always gives exact values when $n = \frac{1}{2}$.

As an application, take the cubic equation,

$$x^3 + 12x = 1{,}120.$$

Here $A = 2$, $B = 1{,}120$, and $C = \sqrt{281} - 1$. Hence, from (1), $x = 15.7$; from (2), $x = 8.4$, and from (3), $x = 10.03$. Thus (3) gives a very close approximation.

Remark by the Editors.

The proposer does not answer (3), (4), and (5). The deductions in his solution result from the confusion of letters representing different quantities and using them to represent the same quantity without drawing the necessary conclusion therefrom. Thus, in the expression $\frac{1}{3}\pi x^2 \times (3r - x)$, the r here is the radius of the sphere from which the segment is taken, whereas in $\frac{1}{2}\pi r^2 x + \frac{1}{6}\pi x^3$, the r is the radius of the base of the segment. If these two expressions are equated as they are in the solution above, it follows that $x = r$ and therefore $n = \frac{1}{2}$. It thus follows that when $n = \frac{1}{2}$ the formula gives correct values for cubics of the proposed form. This seems to us to answer the remaining questions in his problem.

387. Proposed by H. C. FEEMSTER, York College, Nebraska.

Sum the following series:

(1) $$\sum_{n=1}^{n=\infty} \frac{1}{8^{n+1}n!}; \quad (2) \sum_{n=1}^{n=\infty} \frac{1\cdot 3\cdot 5 \cdots 2n+1}{2^{2n+1}(2n-1)!}; \quad (3) \sum_{n=1}^{n=\alpha} \frac{1}{2^{2n}n!}.$$

Solution by J. A. Bullard, Worcester Polytechnic Institute.

By Maclaurin's Theorem, we have

$$f(x) = \sum_{n=0}^{n=\infty} \frac{x^n}{n!} D^n f(0), \qquad (A)$$

where $D^n f(0) = [D^n f(x)]_{x=0}$, $0! = 1$ and $D^0 f(0) = f(0)$.